The Mathematics of Juggling

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Goals of the Talk

▶ Learn a little about juggling.
▶ Learn a little mathematics.
▶ Have some fun.
Reference

Most of the material for this talk was gleaned from Burkard Polster’s great little book *The Mathematics of Juggling*.
What is Juggling?

Juggling takes many forms...
What is Juggling?

Examples

What is Juggling?

Diablo

Tony Frebourg - http://historicaljugglingprops.com/
What is Juggling?

Flair Bartending

Ami Shroff - https://www.facebook.com/amibehramshroff
What is Juggling?

Contact Juggling

Michael Moschen - http://www.michaelmoschen.com/
What is Juggling?

Bounce Juggling

Dan Menendez - http://pianojuggler.com/
What is Juggling?

Juggling is...

- Juggling is manipulating more objects than hands you are using.


- We are interested in *toss juggling*.
What is Juggling?

Toss Juggling Rings

Anthony Gatto - http://www.anthonygatto.com/
What is Juggling?

Toss Juggling Clubs

Jason Garfield - http://jasongarfield.com/
What is Juggling?

Toss Juggling Knives

Edward Gosling - http://chivaree.co.uk/
What is Juggling?

Toss Juggling (Passing)

What is Juggling?

Toss Juggling

We will focus on a single person toss juggling with two hands.

Jason Garfield - http://jasongarfield.com/
Simplifications

Distractions

We are only interested in the various patterns of throws and catches.
We will ignore:

- Showmanship (costumes, unicycles, balancing...)
- Props (kerchiefs, balls, rings, clubs, knives, bowling balls, torches, chainsaws...)
- Variations (inside, outside, piston two-in-one-hand...)
- Flourishes (pirouettes, throwing under the leg or behind the back...)
Simplifying Assumptions

- The balls are juggled at a constant rate, throws alternating between right and left hands.
- Patterns are periodic (otherwise it is not a pattern).
- At most one ball gets caught and thrown on every beat. If a ball is caught, it is the one which is thrown on that beat. (We ignore the time between catch and throw).

These are simple, asynchronous juggling patterns. At the end of the talk we will mention some generalizations.
We use numbers to denote throws.

A throw of height $n$ lands $n$ beats later.

If $n$ is odd it changes hands.

If $n$ is even it lands in the same hand it was thrown from.

A *juggling pattern* is denoted by a string of numbers.

The constant string $n$ is the standard way to juggle $n$ balls.
Representing Juggling Patterns

Juggling Patterns

The constant string $n$ is the standard way to juggle $n$ balls.

- For odd $n$ this is called a *cascade*.
- For even $n$ this is called a *fountain* – and you actually just juggle $n/2$ balls in each hand independently.

- The circular pattern people think of when they think of juggling is called a *shower*. Since this pattern is asymmetric and requires throwing higher, is actually a bad way to start learning.
The throws 0, 1 and 2 are a bit unusual.

- You juggle zero balls by doing nothing (sometimes performers use this opportunity to clap or pirouette or catch a prop thrown by someone else).
- You juggle one ball by fast horizontal throws back and forth.
- You juggle two balls by just holding them (or very small vertical throws, but more often this is an opportunity for flourishes, or biting an apple).
Humans can only throw so high accurately, even if ceilings and wind aren’t an issue.

When you start even 5 level throws are hard. I can manage about a 7.

The world records are around 13.

Thus it makes sense we can cap our throws to a finite level.
Two-Ball Patterns

- 31 – the 2-ball shower. A bad habit.
- 330 – training for 3 balls.
- 40 – two in one hand - training for 4 balls.
- 501 – the interesting and fairly challenging 2-ball pattern!
Examples

Three-Ball Patterns

- 3 – lots of variations, inside/outside/tennis etc.
- 42 – two in one hand. A good opportunity for showmanship.
- 423 – a good first trick, time to bite the apple.
- 522 – how you juggle when you learn.
- 55500 – the flash - add a clap or pirouette to impress.
- 51 – the shower - be sure to practice both ways.
Examples

More Three-Ball Patterns

- 504 – kind of weird.
- 441 – the asynchronous box.
- 4414413 – my favorite, fairly easy to do but impressive looking.
- 50505 – the snake - good training for 5.
- 531 – very pretty when done properly.
Four-Ball Patterns

- 4 – lots of variations: inside/outside/pistons, etc.
- 552 – kind of ugly, not great practice for 5.
- 55550 – better practice for 5.
- 53 – the four-ball half-shower.
- 71 – the four-ball shower.
- 5551 – fun.
- 534, 7531, 7131, 633, 741...
Five-Ball Patterns

- 5
- 73 – the five-ball half-shower.
- 91 – the five-ball shower.
- 64 – three in one hand, two in the other.
- 645, 771, 726, 66661, 663, 744, 77731...
Questions

What strings of natural numbers are valid juggling patterns?
How can we generate all valid juggling patterns?
How many balls are needed to juggle a pattern?
Observations

- The *period* of a pattern is the minimal length before it repeats: $33333 = 3$ has period 1, $441441 = 441$ has period 3.
- Cyclic permutations give the same pattern up to chirality (51 vs. 15 changes direction of the shower).
- Not all strings of natural numbers give valid juggling patterns. We can never have $n(n - 1)$ for example:

![Diagram showing two balls colliding in a juggling pattern]
Let’s tackle the question: given a valid juggling pattern, how many balls are needed to juggle it?

- We let \( t(i) \) be the height of the throw on the \( i \)\textsuperscript{th} beat. That ball lands on the \( i + t(i) \)\textsuperscript{th} beat.
- We assume this is a periodic function with some period \( p \): \( t(i + p) = t(i) \).
- We want to calculate \( \text{balls}(t) \), the number of balls needed to juggle this pattern.
- We define \( \sigma(i) = i + t(i) - \text{balls}(t) \).
- This will be a permutation of \( \mathbb{Z} \) exactly when \( t(i) \) is a valid juggling pattern.
- Conversely, given such a permutation we can recover \( t(i) \).
### A Theorem

### An Example

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<th>-2</th>
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<td>i+t(i)</td>
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<td>σ(i)</td>
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</table>
A Theorem

More Notation

- $\text{height}(t) = \max_i t(i)$ is the maximum throw height that occurs (again we assume this is finite).

- $\text{balls}(t)$, the number of balls needed to juggle the pattern $t$, corresponds to the number of infinite orbits in our diagram (singleton orbits correspond to zero throws, and there are no other possibilities since balls cannot vanish or appear).
Theorem (Average Theorem)

- The number of balls needed to juggle a valid pattern is the average height of the throws.
- If $t$ is a valid juggling pattern with finite height, then

$$\text{balls}(t) = \lim_{|I| \to \infty} \frac{\sum_{i \in I} t(i)}{|I|}$$

Where $I = \{a, a + 1, \ldots, b\}$ is an integer interval and $|I| = b - a + 1$ is the number of elements in that interval.
A Theorem

The Proof

Let $I$ be an integer interval with $|I| > \text{height}(t)$. Then each ball lands at least once in the interval $I$. Let $\mathcal{O}$ be an infinite orbit.
A Theorem

Proof Continued

From the diagram, we can see that

$$|I| - \text{height}(t) \leq \sum_{i \in I \cap O} t(i) \leq |I| + \text{height}(t)$$

Summing over all the orbits (singleton orbits do not contribute because then $t(i) = 0$), we get

$$\text{balls}(t)[|I| - \text{height}(t)] \leq \sum_{i \in I} t(i) \leq \text{balls}(t)[|I| + \text{height}(t)]$$

Dividing by $|I|$ and taking the limit as $|I| \to \infty$ we get the desired result: $\text{balls}(t) = \lim_{|I| \to \infty} \frac{\sum_{i \in I} t(i)}{|I|}$

$\square$
A Theorem

Applications

- **Corollary:** If the average of a string of natural numbers is not an integer, then that string is not a valid juggling pattern.

- **Example:** 5342 is not a valid sequence because \((5 + 3 + 4 + 2)/4 = 15/4\) is not an integer.

- This averaging-to-an-integer condition is not sufficient: 543 is not a valid juggling pattern, but \((5 + 4 + 3)/3 = 4\)

- However, 534 is a valid juggling pattern.

- We do have a partial converse due to Hall:
  **Theorem:** If a string of natural numbers has an integer average, then some permutation of it is a valid juggling pattern.
Note that $\sigma(i + p) = (i + p) + t(i + p) - \text{balls}(t) = i + t(i) - \text{balls}(t) + p = \sigma(i) + p$

$$\sum_{i=1}^{p} (\sigma(i) - i) = \sum_{i=1}^{p} (t(i) - \text{balls}(t)) = p \cdot \text{balls}(t) - p \cdot \text{balls}(t) = 0$$

Recall the affine Weyl group $\tilde{A}_{p-1}$ can be represented as bijective functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ so that $f(i + p) = f(i) + p$, and $\sum_{i=1}^{p} (f(i) - i) = 0$.

Thus the permutations coming from juggling patterns are a subgroup of the affine Weyl group $\tilde{A}_{p-1}$.

In fact, Ehrenborg and Readdy used juggling to calculate the Poincaré series of the affine Weyl groups!
More Results

More Questions

- How can we generate all of the valid juggling patterns? How many are there?
- **Site Swaps:** We switch the landing sites of two different throws to get a new valid pattern with the same number of balls.
Site Swap Example

FIGURE 2.10. Swapping (landing) sites of the balls thrown on beats $i$ and $i+2$ before and after the swap.

FIGURE 2.11. Swapping (landing) sites that both belong to the same orbit.
More Algorithms

If our pattern is not the constant pattern, then there must be a throw which is higher than the average and other throw which is lower than the average. By swapping the landing sites of a higher throw with a lower throw, we get a pattern which is closer to a constant pattern.

Since the throw heights and period are finite, in finitely many steps we must be able to flatten our valid sequence to the constant sequence with the same number of balls.

The Flattening Algorithm: By applying site swaps and cycling permutations, we can take any valid juggling pattern with $b$ balls to the constant sequence $b$. 

Since site swaps and cyclic permutations are reversible, these operations actually generate all the valid juggling patterns with a given period and number of balls!

**Theorem:** The constant pattern consisting of the throw $b$ repeated $p$ times can be transformed into any valid juggling pattern of length $p$ for $b$ balls via site swaps and cyclic permutations.
Examples

- The 3-ball period 2 patterns.

- The 3-ball period 3 patterns.

- Mathematics brought new patterns into juggler’s repertoires!
How many valid juggling patterns are there?

**Theorem:** The number of all minimal $b$-ball juggling patterns (modulo cyclic permutations) of period $p$ is

$$N(b, p) = \frac{1}{p} \sum_{d | p} \mu \left( \frac{p}{d} \right) \left( (b + 1)^d - b^d \right)$$

Where $\mu$ is the Möbius function

$$\mu(n) = \begin{cases} 
0, & \text{if } n \text{ has repeated prime factors} \\
1, & \text{if } n = 1 \text{ or } n \text{ has an even number of prime factors} \\
-1, & \text{if } n \text{ has an odd number of prime factors}
\end{cases}$$
Final Thoughts

Generalizations

- Synchronous patterns – throw and catch with both hands simultaneously.
- Multiple hands – allow for passing between people.
- Multiplexing – throw and catch multiple balls in the same hand at the same time.
- Braids – attach each ball to a string then tie other ends together. Then patterns correspond to braids!
Final Thoughts

- Math is fun.
- Juggling is fun.
- Try playing with http://www.siteswap.net/JsJuggle.html

Thank You For Listening!